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A Fractional Version of Corrected Dual-Simpson's Type Inequality via $s-$convex Function with Applications

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Abstract

Convexity plays a crucial role in mathematical analysis, offering profound insights into the behavior of functions and geometric shapes. Fractional integral operators generalize the classical concept of integration to non-integer orders. In this paper, we establish a new identity by using the Caputo–Fabrizio fractional integral operator. Then by using this new identity, we obtain the corrected dual Simpson's type inequalities for $s-$convex functions. By employing the well-known integral inequalities such as the Hölder's inequality and power-mean inequality, we obtain new error estimates. Furthermore, we discuss the applications to some special means and quadrature formula.

Keywords: corrected dual-Simpson's type inequality; $s-$convex function; fractional integrals; Hölder's inequality; power-mean inequality.

1 Introduction

Convexity is an efficient and useful way to obtain several problems from different branches of science, many researchers have investigated Hermite-Hadamard [20] and Simpson's-type inequalities [13] in the case of a convex function. Indeed, the connection between convexity and integral inequalities has been a subject of significant research and has led to the development of various papers exploring different aspects of convex functions. Akdemir et al. [3] employed a new identity to show several novel integral inequalities for convex functions. Arslan et al. [16] established several novel fractional Hermite-Hadamard and Simpson's-type integral inequalities for strongly (α, m)-convex functions utilizing Caputo–Fabrizio integrals operator. Zhang presented an identical case in [25], as result of geometrically convex functions. Xi [24] obtained the Hermite-Hadamard type inequalities for m and (α, m)-convex functions. We start our work with the following basic definitions.

Definition 1.1. [15] *A function $f : I \to \mathbb{R}$ is said to be convex if,*

$$f\big(\lambda\vartheta_1 + (1-\lambda)\,\vartheta_2\big) \le \lambda f(\vartheta_1) + (1-\lambda)\, f(\vartheta_2),$$

for all $\vartheta_1, \vartheta_2 \in I$ and $\lambda \in [0,1]$.

Definition 1.2. [14] *A function $f : I \subseteq \mathbb{R} \to \mathbb{R}_0 = [0,\infty)$ is said to be $s-$convex if,*

$$f\big(\lambda\vartheta_1 + (1-\lambda)\,\vartheta_2\big) \le \lambda^s f(\vartheta_1) + (1-\lambda)^s\, f(\vartheta_2),$$

for some $s \in (0,1]$ where $\vartheta_1, \vartheta_2 \in I$, $\lambda \in [0,1]$.

The theory of inequality, one of the cornerstones of mathematics, is used in many fields of science. Midpoint-type and Trapezoid-type inequalities formed by the right and left sides of Hermite–Hadamard-type inequalities, Simpson-type inequalities, and Bullen-type inequalities have brought solutions to numerous important studies in the literature. The classical Hermite–Hadamard and Simpson's type inequalities are stated as follows [11],

$$f\left(\frac{\vartheta_1+\vartheta_2}{2}\right) \le \frac{1}{\vartheta_2-\vartheta_1}\int_{\vartheta_1}^{\vartheta_2} f(x)dx \le \frac{f(\vartheta_1)+f(\vartheta_2)}{2}.$$

Theorem 1.1. *A function $f : [\vartheta_1, \vartheta_2] \to \mathbb{R}$ be a four times continuously differentiable function on $(\vartheta_1, \vartheta_2)$ and $\|f^4\|_\infty = \sup_{x\in(\vartheta_1,\vartheta_2)} |f^4(x)|$, then the inequality as follows,*

$$\left[\frac{1}{6} f(\vartheta_1) + \frac{4}{6} f\left(\frac{\vartheta_1+\vartheta_2}{2}\right) + \frac{1}{6} f(\vartheta_2) - \frac{1}{\vartheta_2-\vartheta_1}\int_{\vartheta_1}^{\vartheta_2} f(x)\,dx\right] \le \frac{(\vartheta_2-\vartheta_1)^4}{2880}\|f^4\|_\infty.$$

Recent years, numerous authors have focused on the generalization of Simpson's type inequalities in various kinds of mappings. Some mathematicians discussed the results related to Simpson's and Newton's type in order to obtain a convex map. In particular, Dragomir et al. [8] introduced the most recent Simpson's inequalities for convex function. Additionally, Alomari et al. [4] established few Simpson's type inequalities for $s-$convex functions. Sarikaya et al. [19] identified the significance of the dependence of the variance of the Simpson's type inequality on convexity. Rashid et al. [2] presented some new Simpson's inequalities for generalized $p-$convex function on fractal sets. Fatih et al. [12] proved some fractional Simpson's type inequalities for twice differentiable functions using Riemann–Liouville fractional integral operator. Tariq et al. [23] obtained Simpson's–Mercer inequalities, including Atangana–Baleanu fractional operators.

Nie et al. [17] discussed Simpson's type integral inequalities for $k-$fractional integrals. Du et al. [9] demonstrated the Simpson's-like type inequalities using $m-$convexity. Furthermore, Abdeljawad et al. [1] derived the result general using $(s, m)-$convexity. Set et al. [22] presented the Simpson's-type inequalities for quasi convex functions. The Simpson's-type inequality for generalized convex functions was also established by Sarikaya et al. [18]. Butt et al. [5] proved some Newton type inequalities and Hermite–Hadamard type inequalities [6]. The authors provided the corrected dual-Simpson's formula as follows [10],

$$\int_{\vartheta_1}^{\vartheta_2} f(x)\, dx = S(f) + R(f),$$

where

$$S(f) = \frac{(\vartheta_2 - \vartheta_1)}{15}\left(8f\left(\frac{3\vartheta_1 + \vartheta_2}{4}\right) - f\left(\frac{\vartheta_1 + \vartheta_2}{2}\right) + 8f\left(\frac{\vartheta_1 + 3\vartheta_2}{4}\right)\right),$$

and $R(f)$ show the associated approximation error. Fractional calculus has grown in popularity and relevance over the last three decades, owing to its demonstrated applications in a wide range of seemingly disparate domains of science and engineering. It does give some potentially valuable tools for solving differential and integral equations, a variety of other problems involving mathematical physics special functions, and their extensions and generalizations in one or more variables. The concept of fractional calculus is widely thought to have originated with a question posed to Gottfried Wilhelm Leibniz $(1646 - 1716)$ by Marquis de L'Hôpital $(1661 - 1704)$ in 1695, in which he tried to understand the meaning of Leibniz's notation dy^n/dx^n for the derivative of order $n = \{0, 1, 2, \ldots\}$, when $n = 1/2$ (What if $n = 1/2$?). Leibniz replied to L'Hôpital on 30 September 1695, with the following message: This is an apparent paradox from which, one day, useful consequences will be drawn. Researchers also focused on developing novel transformation techniques that involved fractional operators, they introduced new operators with general and strong kernels that different of locality and singularity, such as the Atangana–Baleanu, Modified Riemann–Liouville, and the Caputo–Fabrizio. Many scholars have introduced several fractional derivative and integral operators that have developed fractional analysis and demonstrated their worth in numerous research domains and presented applications for fractional operators [21].

Definition 1.3. [7] *Let $H^1(\vartheta_1, \vartheta_2)$ be the Sobolev space of order one defined as,*

$$H^1(\vartheta_1, \vartheta_2) = \left\{g \in L^2(\vartheta_1, \vartheta_2) : g' \in L^2(\vartheta_1, \vartheta_2)\right\},$$

where

$$L^2(\vartheta_1, \vartheta_2) = \left\{g(z) : \left(\int_{\vartheta_1}^{\vartheta_2} g^2(z)\, dz\right)^{\frac{1}{2}} < \infty\right\}.$$

Let $f \in H^1(\vartheta_1, \vartheta_2)$, $\vartheta_1 < \vartheta_2$, $\alpha \in [0, 1]$, then the notion of left derivative in the sense of Caputo–Fabrizio is defined as,

$$\left({}_{\vartheta_1}^{CFD}D^{\alpha} f\right)(x) = \frac{\beta(\alpha)}{1-\alpha}\int_{\vartheta_1}^{x} f'(\lambda)\, e^{\frac{-\alpha(x-\lambda)^{\alpha}}{1-\alpha}}\, d\lambda,$$

where $x > \vartheta_1$ and the associated integral operator is

$$\left({}_{\vartheta_1}^{CF}I^{\alpha} f\right)(x) = \frac{1-\alpha}{\beta(\alpha)} f(x) + \frac{\alpha}{\beta(\alpha)}\int_{\vartheta_1}^{x} f(\lambda)\, d\lambda,$$

where $\beta(\alpha) > 0$ is the normalization function satisfying $\beta(0) = \beta(1) = 1$. For $\alpha = 0$ and $\alpha = 1$, the left derivative is defined as follows, respectively,

$$\left({}_{\vartheta_1}^{CFD}D^0 f\right)(x) = f'(x),$$
$$\left({}_{\vartheta_1}^{CFD}D^1 f\right)(x) = f(x) - f(\vartheta_1).$$

For the right derivative operator,

$$\left({}_{\vartheta_2}^{CFD}D^\alpha f\right)(x) = \frac{\beta(\alpha)}{1-\alpha}\int_x^{\vartheta_2} f'(\lambda)\, e^{\frac{-\alpha(\lambda-x)^\alpha}{1-\alpha}}\, d\lambda,$$

where $x < \vartheta_2$ and the associated integral operator is

$$\left({}^{CF}I_{\vartheta_2}^{\alpha} f\right)(x) = \frac{1-\alpha}{\beta(\alpha)} f(x) + \frac{\alpha}{\beta(\alpha)}\int_x^{\vartheta_2} f(\lambda)\, d\lambda,$$

where $\beta(\alpha) > 0$ is a normalization function that satisfies $\beta(0) = \beta(1) = 1$.

In this study, we have introduced the notion of the corrected dual Simpson's type inequalities using Caputo–Fabrizio integrals operator. Firstly, we have established an integral identity that includes the Caputo–Fabrizio integral operator. With the aid of this integral identity, some new corrected dual Simpson's type inequalities for $s-$convex functions are obtained. Lastly, a few applications to special means and quadrature formula are provided.

2 Main Results

For the first time in the inequality theory literature, a corrected dual Simpson's type inequalities for Caputo–Fabrizio fractional integrals operator is given by the following lemma and this lemma constitutes the main motivation of the study.

Lemma 2.1. *Let $f : I \to \mathbb{R}$ be a differentiable function on I^o, $\vartheta_1, \vartheta_2 \in I^o$ with $\vartheta_1 < \vartheta_2$, and $f' \in L_1[\vartheta_1, \vartheta_2]$, then the following equality holds,*

$$\begin{aligned}
&\frac{8}{15} f\left(\frac{3\vartheta_1 + \vartheta_2}{4}\right) - \frac{1}{15} f\left(\frac{\vartheta_1 + \vartheta_2}{2}\right) + \frac{8}{15} f\left(\frac{\vartheta_1 + 3\vartheta_2}{4}\right) \\
&\quad - \frac{\beta(\alpha)}{\alpha(\vartheta_2 - \vartheta_1)}\Big[\left({}_{\vartheta_1}^{CF}I^\alpha f\right)(k) + \left({}^{CF}I_{\vartheta_2}^{\alpha} f\right)(k)\Big] + \frac{2(1-\alpha)}{\beta(\alpha)} f(k) \\
&\qquad = \frac{(\vartheta_2 - \vartheta_1)}{16}\left[\int_0^1 \lambda f'\left((1-\lambda)\vartheta_1 + \lambda\left(\frac{3\vartheta_1 + \vartheta_2}{4}\right)\right) d\lambda\right. \\
&\qquad\quad + \int_0^1 \left(\lambda - \frac{17}{15}\right) f'\left((1-\lambda)\left(\frac{3\vartheta_1 + \vartheta_2}{4}\right) + \lambda\left(\frac{\vartheta_1 + \vartheta_2}{2}\right)\right) d\lambda \\
&\qquad\quad + \int_0^1 \left(\lambda + \frac{2}{15}\right) f'\left((1-\lambda)\left(\frac{\vartheta_1 + \vartheta_2}{2}\right) + \lambda\left(\frac{\vartheta_1 + 3\vartheta_2}{4}\right)\right) d\lambda \\
&\qquad\quad \left. + \int_0^1 (\lambda - 1) f'\left((1-\lambda)\left(\frac{\vartheta_1 + 3\vartheta_2}{4}\right) + \lambda\vartheta_2\right) d\lambda\right],
\end{aligned}$$

where $\beta(\alpha) > 0$ is the normalization function.

Proof. Let

$$\begin{aligned}
&\int_0^1 \lambda f'\left((1-\lambda)\vartheta_1 + \lambda\left(\frac{3\vartheta_1+\vartheta_2}{4}\right)\right)d\lambda \\
&\quad + \int_0^1 \left(\lambda - \frac{17}{15}\right) f'\left((1-\lambda)\left(\frac{3\vartheta_1+\vartheta_2}{4}\right) + \lambda\left(\frac{\vartheta_1+\vartheta_2}{2}\right)\right)d\lambda \\
&\quad + \int_0^1 \left(\lambda + \frac{2}{15}\right) f'\left((1-\lambda)\left(\frac{\vartheta_1+\vartheta_2}{2}\right) + \lambda\left(\frac{\vartheta_1+3\vartheta_2}{4}\right)\right)d\lambda \qquad (1)\\
&\quad + \int_0^1 (\lambda - 1) f'\left((1-\lambda)\left(\frac{\vartheta_1+3\vartheta_2}{4}\right) + \lambda\vartheta_2\right)d\lambda \\
&= I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

Integration by parts, we have

$$\begin{aligned}
I_1 &= \int_0^1 \lambda f'\left((1-\lambda)\vartheta_1 + \lambda\left(\frac{3\vartheta_1+\vartheta_2}{4}\right)\right)d\lambda \\
&= \frac{4}{\vartheta_2-\vartheta_1}\ \lambda f\left((1-\lambda)\vartheta_1 + \lambda\left(\frac{3\vartheta_1+\vartheta_2}{4}\right)\right)\Bigg|_0^1 \qquad (2)\\
&\quad - \frac{4}{\vartheta_2-\vartheta_1}\int_0^1 f\left((1-\lambda)\vartheta_1 + \lambda\left(\frac{3\vartheta_1+\vartheta_2}{4}\right)\right)d\lambda \\
&= \frac{4}{\vartheta_2-\vartheta_1} f\left(\frac{3\vartheta_1+\vartheta_2}{4}\right) - \frac{4}{\vartheta_2-\vartheta_1}\int_0^1 f\left((1-\lambda)\vartheta_1 + \lambda\left(\frac{3\vartheta_1+\vartheta_2}{4}\right)\right)d\lambda \\
&= \frac{4}{\vartheta_2-\vartheta_1} f\left(\frac{3\vartheta_1+\vartheta_2}{4}\right) - \frac{16}{(\vartheta_2-\vartheta_1)^2}\int_{\vartheta_1}^{\frac{3\vartheta_1+\vartheta_2}{4}} f(u)\,du. \qquad (3)
\end{aligned}$$

Similarly, we get

$$\begin{aligned}
I_2 &= \int_0^1 \left(\lambda - \frac{17}{15}\right) f'\left((1-\lambda)\left(\frac{3\vartheta_1+\vartheta_2}{4}\right) + \lambda\left(\frac{\vartheta_1+\vartheta_2}{2}\right)\right)d\lambda \\
&= \frac{4}{\vartheta_2-\vartheta_1}\left(\lambda - \frac{17}{15}\right) f\left((1-\lambda)\left(\frac{3\vartheta_1+\vartheta_2}{4}\right) + \lambda\left(\frac{\vartheta_1+\vartheta_2}{2}\right)\right)\Bigg|_0^1 \\
&\quad - \frac{4}{\vartheta_2-\vartheta_1}\int_0^1 f\left((1-\lambda)\left(\frac{3\vartheta_1+\vartheta_2}{4}\right) + \lambda\left(\frac{\vartheta_1+\vartheta_2}{2}\right)\right)d\lambda \\
&= \frac{-8}{15(\vartheta_2-\vartheta_1)} f\left(\frac{\vartheta_1+\vartheta_2}{2}\right) + \frac{68}{15(\vartheta_2-\vartheta_1)} f\left(\frac{3\vartheta_1+\vartheta_2}{4}\right) \\
&\quad - \frac{4}{\vartheta_2-\vartheta_1}\int_0^1 f\left((1-\lambda)\left(\frac{3\vartheta_1+\vartheta_2}{4}\right) + \lambda\left(\frac{\vartheta_1+\vartheta_2}{2}\right)\right)d\lambda \\
&= \frac{-8}{15(\vartheta_2-\vartheta_1)} f\left(\frac{\vartheta_1+\vartheta_2}{2}\right) + \frac{68}{15(\vartheta_2-\vartheta_1)} f\left(\frac{3\vartheta_1+\vartheta_2}{4}\right) \\
&\quad - \frac{16}{(\vartheta_2-\vartheta_1)^2}\int_{\frac{3\vartheta_1+\vartheta_2}{4}}^{\frac{\vartheta_1+\vartheta_2}{2}} f(u)\,du. \qquad (4)
\end{aligned}$$

$$
\begin{aligned}
I_3 &= \int_0^1 \left(\lambda + \frac{2}{15}\right) f'\left((1-\lambda)\left(\frac{\vartheta_1+\vartheta_2}{2}\right) + \lambda\left(\frac{\vartheta_1+3\vartheta_2}{4}\right)\right) d\lambda \\
&= \frac{4}{\vartheta_2-\vartheta_1} \left(\lambda + \frac{2}{15}\right) f\left((1-\lambda)\left(\frac{\vartheta_1+\vartheta_2}{2}\right) + \lambda\left(\frac{\vartheta_1+3\vartheta_2}{4}\right)\right)\Bigg|_0^1 \\
&\quad - \frac{4}{\vartheta_2-\vartheta_1} \int_0^1 f\left((1-\lambda)\left(\frac{\vartheta_1+\vartheta_2}{2}\right) + \lambda\left(\frac{\vartheta_1+3\vartheta_2}{4}\right)\right) d\lambda \\
&= \frac{68}{15(\vartheta_2-\vartheta_1)} f\left(\frac{\vartheta_1+3\vartheta_2}{4}\right) - \frac{8}{15(\vartheta_2-\vartheta_1)} f\left(\frac{\vartheta_1+\vartheta_2}{2}\right) \\
&\quad - \frac{4}{\vartheta_2-\vartheta_1} \int_0^1 f\left((1-\lambda)\left(\frac{\vartheta_1+\vartheta_2}{2}\right) + \lambda\left(\frac{\vartheta_1+3\vartheta_2}{4}\right)\right) d\lambda \\
&= \frac{68}{15(\vartheta_2-\vartheta_1)} f\left(\frac{\vartheta_1+3\vartheta_2}{4}\right) - \frac{8}{15(\vartheta_2-\vartheta_1)} f\left(\frac{\vartheta_1+\vartheta_2}{2}\right) \\
&\quad - \frac{16}{(\vartheta_2-\vartheta_1)^2} \int_{\frac{\vartheta_1+\vartheta_2}{2}}^{\frac{\vartheta_1+3\vartheta_2}{4}} f(u)\, du,
\end{aligned} \tag{5}
$$

and we have

$$
\begin{aligned}
I_4 &= \int_0^1 (\lambda - 1) f'\left((1-\lambda)\left(\frac{\vartheta_1+3\vartheta_2}{4}\right) + \lambda\vartheta_2\right) d\lambda \\
&= \frac{4}{\vartheta_2-\vartheta_1} (\lambda - 1) f\left((1-\lambda)\left(\frac{\vartheta_1+3\vartheta_2}{4}\right) + \lambda\vartheta_2\right)\Bigg|_0^1 \\
&\quad - \frac{4}{\vartheta_2-\vartheta_1} \int_0^1 f\left((1-\lambda)\left(\frac{\vartheta_1+3\vartheta_2}{4}\right) + \lambda\vartheta_2\right) d\lambda \\
&= \frac{4}{\vartheta_2-\vartheta_1} f\left(\frac{\vartheta_1+3\vartheta_2}{4}\right) - \frac{4}{\vartheta_2-\vartheta_1} \int_0^1 f\left((1-\lambda)\left(\frac{\vartheta_1+3\vartheta_2}{4}\right) + \lambda\vartheta_2\right) d\lambda \\
&= \frac{4}{\vartheta_2-\vartheta_1} f\left(\frac{\vartheta_1+3\vartheta_2}{4}\right) - \frac{16}{(\vartheta_2-\vartheta_1)^2} \int_{\frac{\vartheta_1+3\vartheta_2}{4}}^{\vartheta_2} f(u)\, du.
\end{aligned} \tag{6}
$$

Adding the equalities (1), (4), (5) and (6), we obtain

$$\begin{aligned}
&\int_0^1 \lambda f'\left((1-\lambda)\vartheta_1+\lambda\left(\frac{3\vartheta_1+\vartheta_2}{4}\right)\right)d\lambda \\
&\quad+\int_0^1\left(\lambda-\frac{17}{15}\right)f'\left((1-\lambda)\left(\frac{3\vartheta_1+\vartheta_2}{4}\right)+\lambda\left(\frac{\vartheta_1+\vartheta_2}{2}\right)\right)d\lambda \\
&\quad+\int_0^1\left(\lambda+\frac{2}{15}\right)f'\left((1-\lambda)\left(\frac{\vartheta_1+\vartheta_2}{2}\right)+\lambda\left(\frac{\vartheta_1+3\vartheta_2}{4}\right)\right)d\lambda \\
&\quad+\int_0^1(\lambda-1)f'\left((1-\lambda)\left(\frac{\vartheta_1+3\vartheta_2}{4}\right)+\lambda\vartheta_2\right)d\lambda \\
&=\frac{4}{\vartheta_2-\vartheta_1}f\left(\frac{3\vartheta_1+\vartheta_2}{4}\right)-\frac{16}{(\vartheta_2-\vartheta_1)^2}\int_{\vartheta_1}^{\frac{3\vartheta_1+\vartheta_2}{4}}f(u)\,du \\
&\quad+\frac{-8}{15(\vartheta_2-\vartheta_1)}f\left(\frac{\vartheta_1+\vartheta_2}{2}\right)+\frac{68}{15(\vartheta_2-\vartheta_1)}f\left(\frac{3\vartheta_1+\vartheta_2}{4}\right) \\
&\quad-\frac{16}{(\vartheta_2-\vartheta_1)^2}\int_{\frac{3\vartheta_1+\vartheta_2}{4}}^{\frac{\vartheta_1+\vartheta_2}{2}}f(u)\,du+\frac{68}{15(\vartheta_2-\vartheta_1)}f\left(\frac{\vartheta_1+3\vartheta_2}{4}\right)-\frac{8}{15(\vartheta_2-\vartheta_1)}f\left(\frac{\vartheta_1+\vartheta_2}{2}\right) \\
&\quad-\frac{16}{(\vartheta_2-\vartheta_1)^2}\int_{\frac{\vartheta_1+\vartheta_2}{2}}^{\frac{\vartheta_1+3\vartheta_2}{4}}f(u)\,du+\frac{4}{\vartheta_2-\vartheta_1}f\left(\frac{\vartheta_1+3\vartheta_2}{4}\right)-\frac{16}{(\vartheta_2-\vartheta_1)^2}\int_{\frac{\vartheta_1+3\vartheta_2}{4}}^{\vartheta_2}f(u)\,du \\
&=\frac{128}{15(\vartheta_2-\vartheta_1)}f\left(\frac{3\vartheta_1+\vartheta_2}{4}\right)-\frac{16}{15(\vartheta_2-\vartheta_1)}f\left(\frac{\vartheta_1+\vartheta_2}{2}\right) \\
&\quad+\frac{128}{15(\vartheta_2-\vartheta_1)}f\left(\frac{\vartheta_1+3\vartheta_2}{4}\right)-\frac{16}{(\vartheta_2-\vartheta_1)^2}\int_{\vartheta_1}^{\vartheta_2}f(u)\,du.
\end{aligned} \tag{7}$$

Multiplying the equality (7) with $\frac{(\vartheta_2-\vartheta_1)^2\alpha}{16\beta(\alpha)}$ and subtracting $\frac{2(1-\alpha)}{\beta(\alpha)}f(k)$, we get

$$\begin{aligned}
&\left[\int_0^1 \lambda f'\left((1-\lambda)\vartheta_1+\lambda\left(\frac{3\vartheta_1+\vartheta_2}{4}\right)\right)d\lambda\right. \\
&\quad+\int_0^1\left(\lambda-\frac{17}{15}\right)f'\left((1-\lambda)\left(\frac{3\vartheta_1+\vartheta_2}{4}\right)+\lambda\left(\frac{\vartheta_1+\vartheta_2}{2}\right)\right)d\lambda \\
&\quad+\int_0^1\left(\lambda+\frac{2}{15}\right)f'\left((1-\lambda)\left(\frac{\vartheta_1+\vartheta_2}{2}\right)+\lambda\left(\frac{\vartheta_1+3\vartheta_2}{4}\right)\right)d\lambda \\
&\quad\left.+\int_0^1(\lambda-1)f'\left((1-\lambda)\left(\frac{\vartheta_1+3\vartheta_2}{4}\right)+\lambda\vartheta_2\right)d\lambda\right]\frac{(\vartheta_2-\vartheta_1)^2\alpha}{16\beta(\alpha)}-\frac{2(1-\alpha)}{\beta(\alpha)}f(k) \\
&=\frac{128}{15(\vartheta_2-\vartheta_1)}f\left(\frac{3\vartheta_1+\vartheta_2}{4}\right)\frac{(\vartheta_2-\vartheta_1)^2\alpha}{16\beta(\alpha)}-\frac{16}{15(\vartheta_2-\vartheta_1)}f\left(\frac{\vartheta_1+\vartheta_2}{2}\right)\frac{(\vartheta_2-\vartheta_1)^2\alpha}{16\beta(\alpha)} \\
&\quad+\frac{128}{15(\vartheta_2-\vartheta_1)}f\left(\frac{\vartheta_1+3\vartheta_2}{4}\right)\frac{(\vartheta_2-\vartheta_1)^2\alpha}{16\beta(\alpha)}-\frac{\alpha}{\beta(\alpha)}\int_{\vartheta_1}^{\vartheta_2}f(u)\,du-\frac{2(1-\alpha)}{\beta(\alpha)}f(k)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{8\alpha}{15\beta(\alpha)} f\left(\frac{3\vartheta_1+\vartheta_2}{4}\right) - \frac{\alpha}{15\beta(\alpha)} f\left(\frac{\vartheta_1+\vartheta_2}{2}\right) + \frac{8\alpha}{15\beta(\alpha)} f\left(\frac{\vartheta_1+3\vartheta_2}{4}\right) \\
&\quad - \left(\frac{\alpha}{\beta(\alpha)}\int_{\vartheta_1}^{k} f(u)\,du - \frac{(1-\alpha)}{\beta(\alpha)} f(k) + \frac{\alpha}{\beta(\alpha)}\int_{k}^{\vartheta_2} f(u)\,du - \frac{(1-\alpha)}{\beta(\alpha)} f(k)\right) \\
&= \frac{8\alpha}{15\beta(\alpha)} f\left(\frac{3\vartheta_1+\vartheta_2}{4}\right) - \frac{\alpha}{15\beta(\alpha)} f\left(\frac{\vartheta_1+\vartheta_2}{2}\right) + \frac{8\alpha}{15\beta(\alpha)} f\left(\frac{\vartheta_1+3\vartheta_2}{4}\right) \\
&\quad - \left[\left({}^{CF}_{\vartheta_1}I^{\alpha} f\right)(k) + \left({}^{CF}I^{\alpha}_{\vartheta_2} f\right)(k)\right].
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
&\frac{8}{15} f\left(\frac{3\vartheta_1+\vartheta_2}{4}\right) - \frac{1}{15} f\left(\frac{\vartheta_1+\vartheta_2}{2}\right) + \frac{8}{15} f\left(\frac{\vartheta_1+3\vartheta_2}{4}\right) \\
&\qquad - \frac{\beta(\alpha)}{\alpha(\vartheta_2-\vartheta_1)}\left[\left({}^{CF}_{\vartheta_1}I^{\alpha} f\right)(k) + \left({}^{CF}I^{\alpha}_{\vartheta_2} f\right)(k)\right] + \frac{2(1-\alpha)}{\beta(\alpha)} f(k) \\
&= \frac{(\vartheta_2-\vartheta_1)}{16}\left[\int_0^1 \lambda f'\left((1-\lambda)\vartheta_1 + \lambda\left(\frac{3\vartheta_1+\vartheta_2}{4}\right)\right)d\lambda\right. \\
&\qquad + \int_0^1 \left(\lambda - \frac{17}{15}\right) f'\left((1-\lambda)\left(\frac{3\vartheta_1+\vartheta_2}{4}\right) + \lambda\left(\frac{\vartheta_1+\vartheta_2}{2}\right)\right)d\lambda \\
&\qquad + \int_0^1 \left(\lambda + \frac{2}{15}\right) f'\left((1-\lambda)\left(\frac{\vartheta_1+\vartheta_2}{2}\right) + \lambda\left(\frac{\vartheta_1+3\vartheta_2}{4}\right)\right)d\lambda \\
&\qquad \left. + \int_0^1 (\lambda-1) f'\left((1-\lambda)\left(\frac{\vartheta_1+3\vartheta_2}{4}\right) + \lambda\vartheta_2\right)d\lambda\right].
\end{aligned}
$$

This completes the proof. □

Theorem 2.1. *Under the assumptions of Lemma 2.1. If $|f'|$ is $s-$convex for some fixed $s \in (0,1]$, then the following inequality holds,*

$$
\begin{aligned}
&\left|\frac{8}{15} f\left(\frac{3\vartheta_1+\vartheta_2}{4}\right) - \frac{1}{15} f\left(\frac{\vartheta_1+\vartheta_2}{2}\right) + \frac{8}{15} f\left(\frac{\vartheta_1+3\vartheta_2}{4}\right)\right. \\
&\qquad \left. - \frac{\beta(\alpha)}{\alpha(\vartheta_2-\vartheta_1)}\left[\left({}^{CF}_{\vartheta_1}I^{\alpha} f\right)(k) + \left({}^{CF}I^{\alpha}_{\vartheta_2} f\right)(k)\right] + \frac{2(1-\alpha)}{\beta(\alpha)} f(k)\right| \\
&\leq \frac{(\vartheta_2-\vartheta_1)}{240(s+1)(s+2)}\left[\left(4^{1-s} - 17 + 15\times 2^{2+s} + 15\times 2^{3+2s} - 43\right|\right. \\
&\qquad \left. \times\, 3^{1+s} + \left(-1+3^{1+s}\right)s\right)\Big]\left(|f'(\vartheta_1)| + |f'(\vartheta_2)|\right).
\end{aligned}
$$

Proof. By taking modulus in Lemma 2.1, and with the help of the $s-$convexity of $|f'|$, we get

$$\begin{aligned}
&\left|\frac{8}{15}f\left(\frac{3\vartheta_1+\vartheta_2}{4}\right)-\frac{1}{15}f\left(\frac{\vartheta_1+\vartheta_2}{2}\right)+\frac{8}{15}f\left(\frac{\vartheta_1+3\vartheta_2}{4}\right)\right.\\
&\quad\left.-\frac{\beta(\alpha)}{\alpha(\vartheta_2-\vartheta_1)}\Big[\left({}_{\vartheta_1}^{CF}I^{\alpha}f\right)(k)+\left({}^{CF}I^{\alpha}_{\vartheta_2}f\right)(k)\Big]+\frac{2(1-\alpha)}{\beta(\alpha)}f(k)\right|\\
&\le\frac{(\vartheta_2-\vartheta_1)}{16}\left[\int_0^1|\lambda|\left|f'\left((1-\lambda)\vartheta_1+\lambda\left(\frac{3\vartheta_1+\vartheta_2}{4}\right)\right)\right|d\lambda\right.\\
&\quad+\int_0^1\left|\lambda-\frac{17}{15}\right|\left|f'\left((1-\lambda)\left(\frac{3\vartheta_1+\vartheta_2}{4}\right)+\lambda\left(\frac{\vartheta_1+\vartheta_2}{2}\right)\right)\right|d\lambda\\
&\quad+\int_0^1\left|\lambda+\frac{2}{15}\right|\left|f'\left((1-\lambda)\left(\frac{\vartheta_1+\vartheta_2}{2}\right)+\lambda\left(\frac{\vartheta_1+3\vartheta_2}{4}\right)\right)\right|d\lambda\\
&\quad\left.+\int_0^1|\lambda-1|\left|f'\left((1-\lambda)\left(\frac{\vartheta_1+3\vartheta_2}{4}\right)+\lambda\vartheta_2\right)\right|d\lambda\right]\\
&=\frac{(\vartheta_2-\vartheta_1)}{16}\left[\int_0^1\lambda\left|f'\left(\frac{4-\lambda}{4}\vartheta_1+\frac{\lambda}{4}\vartheta_2\right)\right|d\lambda\right.\\
&\quad+\int_0^1\left(\frac{17}{15}-\lambda\right)\left|f'\left(\frac{3-\lambda}{4}\vartheta_1+\frac{1+\lambda}{4}\vartheta_2\right)\right|d\lambda\\
&\quad+\int_0^1\left(\lambda+\frac{2}{15}\right)\left|f'\left(\frac{2-\lambda}{4}\vartheta_1+\frac{2+\lambda}{4}\vartheta_2\right)\right|d\lambda\\
&\quad\left.+\int_0^1(1-\lambda)\left|f'\left(\frac{1-\lambda}{4}\vartheta_1+\frac{3+\lambda}{4}\vartheta_2\right)\right|d\lambda\right]\\
&=\frac{(\vartheta_2-\vartheta_1)}{16}\left[\int_0^1\lambda\left(\left(\frac{4-\lambda}{4}\right)^s|f'(\vartheta_1)|+\left(\frac{\lambda}{4}\right)^s|f'(\vartheta_2)|\right)d\lambda\right.\\
&\quad+\int_0^1\left(\frac{17}{15}-\lambda\right)\left(\left(\frac{3-\lambda}{4}\right)^s|f'(\vartheta_1)|+\left(\frac{1+\lambda}{4}\right)^s|f'(\vartheta_2)|\right)d\lambda\\
&\quad+\int_0^1\left(\lambda+\frac{2}{15}\right)\left(\left(\frac{2-\lambda}{4}\right)^s|f'(\vartheta_1)|+\left(\frac{2+\lambda}{4}\right)^s|f'(\vartheta_2)|\right)d\lambda\\
&\quad\left.+\int_0^1(1-\lambda)\left(\left(\frac{1-\lambda}{4}\right)^s|f'(\vartheta_1)|+\left(\frac{3+\lambda}{4}\right)^s|f'(\vartheta_2)|\right)d\lambda\right]\\
&=\frac{(\vartheta_2-\vartheta_1)}{240(s+1)(s+2)}\Big[\big(4^{1-s}-17+15\times 2^{2+s}+15\times 2^{3+2s}-43|\\
&\quad\times 3^{1+s}+\left(-1+3^{1+s}\right)s\big)\Big]\Big(|f'(\vartheta_1)|+|f'(\vartheta_2)|\Big).
\end{aligned}$$

This completes the proof. □

Corollary 2.1. *If we choose $s=1$ in Theorem 2.1, we have*

$$\begin{aligned}
&\left|\frac{8}{15}f\left(\frac{3\vartheta_1+\vartheta_2}{4}\right)-\frac{1}{15}f\left(\frac{\vartheta_1+\vartheta_2}{2}\right)+\frac{8}{15}f\left(\frac{\vartheta_1+3\vartheta_2}{4}\right)\right.\\
&\quad\left.-\frac{\beta(\alpha)}{\alpha(\vartheta_2-\vartheta_1)}\Big[\left({}_{\vartheta_1}^{CF}I^{\alpha}f\right)(k)+\left({}^{CF}I^{\alpha}_{\vartheta_2}f\right)(k)\Big]+\frac{2(1-\alpha)}{\beta(\alpha)}f(k)\right|\\
&\le\frac{17(\vartheta_2-\vartheta_1)}{60}\left(\frac{|f'(\vartheta_1)|+|f'(\vartheta_2)|}{2}\right).
\end{aligned}$$

Theorem 2.2. *Under the assumptions of Lemma 2.1. If $|f'|^q$ is $s-$convex for some fixed $s\in(0,1]$ and*

$q > 1$, *then the following inequality holds,*

$$\begin{aligned}
&\left|\frac{8}{15}f\left(\frac{3\vartheta_1+\vartheta_2}{4}\right)-\frac{1}{15}f\left(\frac{\vartheta_1+\vartheta_2}{2}\right)+\frac{8}{15}f\left(\frac{\vartheta_1+3\vartheta_2}{4}\right)\right.\\
&\quad\left.-\frac{\beta(\alpha)}{\alpha(\vartheta_2-\vartheta_1)}\Big[\left({}^{CF}_{\vartheta_1}I^{\alpha}f\right)(k)+\left({}^{CF}I^{\alpha}_{\vartheta_2}f\right)(k)\Big]+\frac{2(1-\alpha)}{\beta(\alpha)}f(k)\right|\\
&\leq\frac{(\vartheta_2-\vartheta_1)}{16}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\Bigg[\left(\left(\frac{4-3^{s+1}\times 4^{-s}}{(s+1)}\right)|f'(\vartheta_1)|^q+\left(\frac{4^{-s}}{(s+1)}\right)|f'(\vartheta_2)|^q\right)^{\frac{1}{q}}\\
&\quad+\left(\frac{17^{p+1}-2^{p+1}}{15^{p+1}}\right)^{\frac{1}{p}}\left(\left(\frac{4^{s}\left(-2^{s+1}+3^{s+1}\right)}{(s+1)}\right)|f'(\vartheta_1)|^q+\left(\frac{4^{-s}\left(-1+2^{s+1}\right)}{(s+1)}\right)|f'(\vartheta_2)|^q\right)^{\frac{1}{q}}\\
&\quad+\left(\frac{17^{p+1}-2^{p+1}}{15^{p+1}}\right)^{\frac{1}{p}}\left(\left(\frac{4^{-s}\left(-1+2^{s+1}\right)}{(s+1)}\right)|f'(\vartheta_1)|^q+\left(\frac{4^{s}\left(-2^{s+1}+3^{s+1}\right)}{(s+1)}\right)|f'(\vartheta_2)|^q\right)^{\frac{1}{q}}\\
&\quad+\left(\left(\frac{4^{-s}}{(s+1)}\right)|f'(\vartheta_1)|^q+\left(\frac{4-3^{s+1}\times 4^{-s}}{(s+1)}\right)|f'(\vartheta_2)|^q\right)^{\frac{1}{q}}\Bigg].
\end{aligned}$$

Proof. By using the Hölder's inequality, and with the help of the $s-$convexity of $|f'|^q$, we get

$$\begin{aligned}
&\left|\frac{8}{15}f\left(\frac{3\vartheta_1+\vartheta_2}{4}\right)-\frac{1}{15}f\left(\frac{\vartheta_1+\vartheta_2}{2}\right)+\frac{8}{15}f\left(\frac{\vartheta_1+3\vartheta_2}{4}\right)\right.\\
&\quad\left.-\frac{\beta(\alpha)}{\alpha(\vartheta_2-\vartheta_1)}\left[\left({}^{CF}_{\vartheta_1}I^{\alpha}f\right)(k)+\left({}^{CF}I^{\alpha}_{\vartheta_2}f\right)(k)\right]+\frac{2(1-\alpha)}{\beta(\alpha)}f(k)\right|\\
&\le\frac{(\vartheta_2-\vartheta_1)}{16}\left[\left(\int_0^1|\lambda|^p\,d\lambda\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(\frac{4-\lambda}{4}\vartheta_1+\frac{\lambda}{4}\vartheta_2\right)\right|^q\right)^{\frac{1}{q}}d\lambda\right.\\
&\quad+\left(\int_0^1\left|\lambda-\frac{17}{15}\right|^p d\lambda\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(\frac{3-\lambda}{4}\vartheta_1+\frac{1+\lambda}{4}\vartheta_2\right)\right|^q\right)^{\frac{1}{q}}d\lambda\\
&\quad+\left(\int_0^1\left|\lambda+\frac{2}{15}\right|^p d\lambda\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(\frac{2-\lambda}{4}\vartheta_1+\frac{2+\lambda}{4}\vartheta_2\right)\right|^q\right)^{\frac{1}{q}}d\lambda\\
&\quad\left.+\left(\int_0^1|\lambda-1|^p\,d\lambda\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(\frac{1-\lambda}{4}\vartheta_1+\frac{3+\lambda}{4}\vartheta_2\right)\right|^q\right)^{\frac{1}{q}}d\lambda\right],\\
&=\frac{(\vartheta_2-\vartheta_1)}{16}\left[\left(\int_0^1(\lambda)^p\,d\lambda\right)^{\frac{1}{p}}\left(\int_0^1\left(\left(\frac{4-\lambda}{4}\right)^s|f'(\vartheta_1)|^q+\left(\frac{\lambda}{4}\right)^s|f'(\vartheta_2)|^q\right)^{\frac{1}{q}}\right)d\lambda\right.\\
&\quad+\left(\int_0^1\left(\frac{17}{15}-\lambda\right)^p d\lambda\right)^{\frac{1}{p}}\left(\int_0^1\left(\left(\frac{3-\lambda}{4}\right)^s|f'(\vartheta_1)|^q+\left(\frac{1+\lambda}{4}\right)^s|f'(\vartheta_2)|^q\right)^{\frac{1}{q}}\right)d\lambda\\
&\quad+\left(\int_0^1\left(\lambda+\frac{2}{15}\right)^p d\lambda\right)^{\frac{1}{p}}\left(\int_0^1\left(\left(\frac{2-\lambda}{4}\right)^s|f'(\vartheta_1)|^q+\left(\frac{2+\lambda}{4}\right)^s|f'(\vartheta_2)|^q\right)^{\frac{1}{q}}\right)d\lambda\\
&\quad\left.+\left(\int_0^1(1-\lambda)^p\,d\lambda\right)^{\frac{1}{p}}\left(\int_0^1\left(\left(\frac{1-\lambda}{4}\right)^s|f'(\vartheta_1)|^q+\left(\frac{3+\lambda}{4}\right)^s|f'(\vartheta_2)|^q\right)^{\frac{1}{q}}\right)d\lambda\right],\\
&=\frac{(\vartheta_2-\vartheta_1)}{16}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\left(\left(\frac{4-3^{s+1}\times4^{-s}}{(s+1)}\right)|f'(\vartheta_1)|^q+\left(\frac{4^{-s}}{(s+1)}\right)|f'(\vartheta_2)|^q\right)^{\frac{1}{q}}\right.\\
&\quad+\left(\frac{17^{p+1}-2^{p+1}}{15^{p+1}}\right)^{\frac{1}{p}}\left(\left(\frac{4^s\left(-2^{s+1}+3^{s+1}\right)}{(s+1)}\right)|f'(\vartheta_1)|^q+\left(\frac{4^{-s}\left(-1+2^{s+1}\right)}{(s+1)}\right)|f'(\vartheta_2)|^q\right)^{\frac{1}{q}}\\
&\quad+\left(\frac{17^{p+1}-2^{p+1}}{15^{p+1}}\right)^{\frac{1}{p}}\left(\left(\frac{4^{-s}\left(-1+2^{s+1}\right)}{(s+1)}\right)|f'(\vartheta_1)|^q+\left(\frac{4^s\left(-2^{s+1}+3^{s+1}\right)}{(s+1)}\right)|f'(\vartheta_2)|^q\right)^{\frac{1}{q}}\\
&\quad\left.+\left(\left(\frac{4^{-s}}{(s+1)}\right)|f'(\vartheta_1)|^q+\left(\frac{4-3^{s+1}\times4^{-s}}{(s+1)}\right)|f'(\vartheta_2)|^q\right)^{\frac{1}{q}}\right].
\end{aligned}$$

This completes the proof. □

Corollary 2.2. *If we choose $s = 1$ in Theorem 2.2, we have*

$$\begin{aligned}
&\left|\frac{8}{15}f\left(\frac{3\vartheta_1+\vartheta_2}{4}\right)-\frac{1}{15}f\left(\frac{\vartheta_1+\vartheta_2}{2}\right)+\frac{8}{15}f\left(\frac{\vartheta_1+3\vartheta_2}{4}\right)\right.\\
&\quad\left.-\frac{\beta(\alpha)}{\alpha(\vartheta_2-\vartheta_1)}\left[\left({}^{CF}_{\vartheta_1}I^{\alpha}f\right)(k)+\left({}^{CF}I^{\alpha}_{\vartheta_2}f\right)(k)\right]+\frac{2(1-\alpha)}{\beta(\alpha)}f(k)\right|\\
&\leq\frac{(\vartheta_2-\vartheta_1)}{16}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\left(\frac{7|f'(\vartheta_1)|^q+|f'(\vartheta_2)|^q}{8}\right)^{\frac{1}{q}}+\left(\frac{|f'(\vartheta_1)|^q+7|f'(\vartheta_2)|^q}{8}\right)^{\frac{1}{q}}\right.\\
&\quad\left.+\left(\frac{17^{p+1}-2^{p+1}}{15^{p+1}}\right)^{\frac{1}{p}}\left(\left(\frac{5|f'(\vartheta_1)|^q+3|f'(\vartheta_2)|^q}{8}\right)^{\frac{1}{q}}+\left(\frac{3|f'(\vartheta_1)|^q+5|f'(\vartheta_2)|^q}{8}\right)^{\frac{1}{q}}\right)\right].
\end{aligned}$$

Theorem 2.3. *Under the assumption of Lemma 2.1. If $|f'|^q$ is $s-$convex for some fixed $s\in(0,1]$ and $q\geq 1$, then the following inequality holds,*

$$\begin{aligned}
&\left|\frac{8}{15}f\left(\frac{3\vartheta_1+\vartheta_2}{4}\right)-\frac{1}{15}f\left(\frac{\vartheta_1+\vartheta_2}{2}\right)+\frac{8}{15}f\left(\frac{\vartheta_1+3\vartheta_2}{4}\right)\right.\\
&\quad\left.-\frac{\beta(\alpha)}{\alpha(\vartheta_2-\vartheta_1)}\left[\left({}^{CF}_{\vartheta_1}I^{\alpha}f\right)(k)+\left({}^{CF}I^{\alpha}_{\vartheta_2}f\right)(k)\right]+\frac{2(1-\alpha)}{\beta(\alpha)}f(k)\right|\\
&\leq\frac{(\vartheta_2-\vartheta_1)}{16}\left[\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\left(\frac{4^s\left(4^{s+2}-3^{s+1}(5+s)\right)}{(s+1)(s+2)}\right)|f'(\vartheta_1)|^q+\left(\frac{4^{-s}}{(s+2)}\right)|f'(\vartheta_2)|^q\right)^{\frac{1}{q}}\right.\\
&\quad+\left(\frac{19}{30}\right)^{1-\frac{1}{q}}\times\left(\left(\frac{4^{-s}\left(-2^{s+2}(-13+s)+3^{s+1}(-11+17s)\right)}{15(s+1)(s+2)}\right)|f'(\vartheta_1)|^q\right.\\
&\quad\left.+\left(\frac{4^{-s}\left(-49-17s+2^{s+2}(7+s)\right)}{15(s+1)(s+2)}\right)|f'(\vartheta_2)|^q\right)^{\frac{1}{q}}\\
&\quad+\left(\frac{19}{30}\right)^{1-\frac{1}{q}}\times\left(\left(\frac{4^{-s}\left(-49-17s+2^{s+2}(7+s)\right)}{15(s+1)(s+2)}\right)|f'(\vartheta_1)|^q\right.\\
&\quad\left.+\left(\frac{4^{-s}\left(-2^{s+2}(-13+s)+3^{s+1}(-11+17s)\right)}{15(s+1)(s+2)}\right)|f'(\vartheta_2)|^q\right)^{\frac{1}{q}}\\
&\quad\left.+\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\left(\frac{4^{-s}}{(s+2)}\right)|f'(\vartheta_1)|^q+\left(\frac{4^s\left(4^{s+2}-3^{s+1}(5+s)\right)}{(s+1)(s+2)}\right)|f'(\vartheta_2)|^q\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Proof. By using the power-mean inequality, and with the help of the $s-$convexity of $|f'|^q$, we get

$$\begin{aligned}
&\left|\frac{8}{15}f\left(\frac{3\vartheta_1+\vartheta_2}{4}\right)-\frac{1}{15}f\left(\frac{\vartheta_1+\vartheta_2}{2}\right)+\frac{8}{15}f\left(\frac{\vartheta_1+3\vartheta_2}{4}\right)\right.\\
&\quad\left.-\frac{\beta(\alpha)}{\alpha(\vartheta_2-\vartheta_1)}\left[\left({}^{CF}_{\vartheta_1}I^{\alpha}f\right)(k)+\left({}^{CF}I^{\alpha}_{\vartheta_2}f\right)(k)\right]+\frac{2(1-\alpha)}{\beta(\alpha)}f(k)\right|\\
&\le\frac{(\vartheta_2-\vartheta_1)}{16}\left[\left(\int_0^1\lambda d\lambda\right)^{1-\frac{1}{q}}\left(\int_0^1\lambda\left|f'\left(\frac{4-\lambda}{4}\vartheta_1+\frac{\lambda}{4}\vartheta_2\right)\right|^q\right)^{\frac{1}{q}}d\lambda\right.\\
&\quad+\left(\int_0^1\left|\lambda-\frac{17}{15}\right|d\lambda\right)^{1-\frac{1}{q}}\left(\int_0^1\left|\lambda-\frac{17}{15}\right|\left|f'\left(\frac{3-\lambda}{4}\vartheta_1+\frac{1+\lambda}{4}\vartheta_2\right)\right|^q\right)^{\frac{1}{q}}d\lambda\\
&\quad+\left(\int_0^1\left|\lambda+\frac{2}{15}\right|d\lambda\right)^{1-\frac{1}{q}}\left(\int_0^1\left|\lambda+\frac{2}{15}\right|\left|f'\left(\frac{2-\lambda}{4}\vartheta_1+\frac{2+\lambda}{4}\vartheta_2\right)\right|^q\right)^{\frac{1}{q}}d\lambda\\
&\quad\left.+\left(\int_0^1|\lambda-1|d\lambda\right)^{1-\frac{1}{q}}\left(\int_0^1|\lambda-1|\left|f'\left(\frac{1-\lambda}{4}\vartheta_1+\frac{3+\lambda}{4}\vartheta_2\right)\right|^q\right)^{\frac{1}{q}}d\lambda\right]\\
&=\frac{(\vartheta_2-\vartheta_1)}{16}\left[\left(\int_0^1\lambda d\lambda\right)^{1-\frac{1}{q}}\left(\int_0^1\lambda\left(\left(\frac{4-\lambda}{4}\right)^s|f'(\vartheta_1)|^q+\left(\frac{\lambda}{4}\right)^s|f'(\vartheta_2)|^q\right)\right)^{\frac{1}{q}}d\lambda\right.\\
&\quad+\left(\int_0^1\left(\frac{17}{15}-\lambda\right)d\lambda\right)^{1-\frac{1}{q}}\left(\int_0^1\left(\frac{17}{15}-\lambda\right)\left(\left(\frac{3-\lambda}{4}\right)^s|f'(\vartheta_1)|^q+\left(\frac{1+\lambda}{4}\right)^s|f'(\vartheta_2)|^q\right)\right)^{\frac{1}{q}}d\lambda\\
&\quad+\left(\int_0^1\left(\lambda+\frac{2}{15}\right)d\lambda\right)^{1-\frac{1}{q}}\left(\int_0^1\left(\lambda+\frac{2}{15}\right)\left(\left(\frac{2-\lambda}{4}\right)^s|f'(\vartheta_1)|^q+\left(\frac{2+\lambda}{4}\right)^s|f'(\vartheta_2)|^q\right)\right)^{\frac{1}{q}}d\lambda\\
&\quad\left.+\left(\int_0^1(1-\lambda)d\lambda\right)^{1-\frac{1}{q}}\left(\int_0^1(1-\lambda)\left(\left(\frac{1-\lambda}{4}\right)^s|f'(\vartheta_1)|^q+\left(\frac{3+\lambda}{4}\right)^s|f'(\vartheta_2)|^q\right)\right)^{\frac{1}{q}}d\lambda\right]\\
&=\frac{(\vartheta_2-\vartheta_1)}{16}\left[\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\left(\frac{4^s\left(4^{s+2}-3^{s+1}(5+s)\right)}{(s+1)(s+2)}\right)|f'(\vartheta_1)|^q+\left(\frac{4^{-s}}{(s+2)}\right)|f'(\vartheta_2)|^q\right)^{\frac{1}{q}}\right.\\
&\quad+\left(\frac{19}{30}\right)^{1-\frac{1}{q}}\times\left(\left(\frac{4^{-s}\left(-2^{s+2}(-13+s)+3^{s+1}(-11+17s)\right)}{15(s+1)(s+2)}\right)|f'(\vartheta_1)|^q\right.\\
&\quad\left.+\left(\frac{4^{-s}\left(-49-17s+2^{s+2}(7+s)\right)}{15(s+1)(s+2)}\right)|f'(\vartheta_2)|^q\right)^{\frac{1}{q}}\\
&\quad+\left(\frac{19}{30}\right)^{1-\frac{1}{q}}\times\left(\left(\frac{4^{-s}\left(-49-17s+2^{s+2}(7+s)\right)}{15(s+1)(s+2)}\right)|f'(\vartheta_1)|^q\right.\\
&\quad\left.+\left(\frac{4^{-s}\left(-2^{s+2}(-13+s)+3^{s+1}(-11+17s)\right)}{15(s+1)(s+2)}\right)|f'(\vartheta_2)|^q\right)^{\frac{1}{q}}\\
&\quad\left.+\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\left(\frac{4^{-s}}{(s+2)}\right)|f'(\vartheta_1)|^q+\left(\frac{4^s\left(4^{s+2}-3^{s+1}(5+s)\right)}{(s+1)(s+2)}\right)|f'(\vartheta_2)|^q\right)^{\frac{1}{q}}\right].
\end{aligned}$$

This completes the proof. □

Corollary 2.3. *If we choose $s = 1$ in Theorem 2.3, we have*

$$\left|\frac{8}{15} f\left(\frac{3\vartheta_1+\vartheta_2}{4}\right)-\frac{1}{15} f\left(\frac{\vartheta_1+\vartheta_2}{2}\right)+\frac{8}{15} f\left(\frac{\vartheta_1+3\vartheta_2}{4}\right)\right.$$
$$\left.-\frac{\beta(\alpha)}{\alpha(\vartheta_2-\vartheta_1)}\left[\left({}_{\vartheta_1}^{CF}I^{\alpha} f\right)(k)+\left({}^{CF}I_{\vartheta_2}^{\alpha} f\right)(k)\right]+\frac{2(1-\alpha)}{\beta(\alpha)} f(k)\right|$$
$$\leq \frac{(\vartheta_2-\vartheta_1)}{16}\left[\left(\frac{5|f'(\vartheta_1)|^q+|f'(\vartheta_2)|^q}{12}\right)+\left(\frac{|f'(\vartheta_1)|^q+5|f'(\vartheta_2)|^q}{12}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{19}{30}\right)^{1-\frac{1}{q}}\left(\left(\frac{300|f'(\vartheta_1)|^q+156|f'(\vartheta_2)|^q}{720}\right)+\left(\frac{156|f'(\vartheta_1)|^q+300|f'(\vartheta_2)|^q}{720}\right)^{\frac{1}{q}}\right)\right].$$

3 Application to Some Special Means

We shall consider the following special means:

(a) The arithmetic mean,

$$A = A(\vartheta_1, \vartheta_2) := \frac{\vartheta_1+\vartheta_2}{2}, \qquad \vartheta_1, \vartheta_2 \in \mathbb{R}.$$

(b) The logarithmic mean,

$$L = L(\vartheta_1, \vartheta_2) := \frac{\vartheta_2-\vartheta_1}{\ln\vartheta_2-\ln\vartheta_1}, \qquad \vartheta_1, \vartheta_2 \in \mathbb{R}, \qquad \vartheta_1 \neq \vartheta_2.$$

(c) The generalized logarithmic-mean,

$$L_r = L_r(\vartheta_1, \vartheta_2) := \left[\frac{\vartheta_2^{r+1}-\vartheta_1^{r+1}}{(r+1)(\vartheta_2-\vartheta_1)}\right], \qquad r \in \mathbb{R}\backslash\{-1, 0\}, \qquad \vartheta_1, \vartheta_2 \in \mathbb{R}, \qquad \vartheta_1 \neq \vartheta_2.$$

Proposition 3.1. *Let $\vartheta_1, \vartheta_2 \in \mathbb{R}$ with $0 < \vartheta_1 < \vartheta_2$, then we have*

$$\left|16A\left(\left(\frac{3\vartheta_1+\vartheta_2}{4}\right),\left(\frac{\vartheta_1+3\vartheta_2}{4}\right)\right)-A^n(\vartheta_1,\vartheta_2)-15L_n^n(\vartheta_1,\vartheta_2)\right|$$
$$\leq \frac{17n(\vartheta_2-\vartheta_1)}{8}\left[(\vartheta_1)^{(n-1)}+(\vartheta_2)^{(n-1)}\right].$$

Proof. The assertion follows from Corollary 2.1, using the function $f(x) = x^n$ and $\alpha = 1$, $\beta(0) = \beta(1) = 1$. ☐

4 Application to Corrected Dual-Simpson's Quadrature F ormula

Let Z is the partition of the points $\vartheta_1 = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = \vartheta_2$ of the interval $[\vartheta_1, \vartheta_2]$, consider the quadrature formula,

$$\int_{\vartheta_1}^{\vartheta_2} f(x)\,dx = \mu(f, Z) + R(f, Z),$$

where

$$\mu\left(f,Z\right)=\sum_{i=0}^{n-1}\left[\frac{\left(x_{i+1}-x_{i}\right)}{15}\left(8f\left(\frac{3x_{i}+x_{i+1}}{4}\right)-f\left(\frac{x_{i}+x_{i+1}}{2}\right)+8f\left(\frac{x_{i}+3x_{i+1}}{4}\right)\right)\right],$$

and $R\left(f,Z\right)$ show the associated approximation error.

Proposition 4.1. *Let $n\in f$ and $f:[\vartheta_1,\vartheta_2]\to\mathbb{R}$ be a differentiable function on $(\vartheta_1,\vartheta_2)$ with $0\leq\vartheta_1<\vartheta_2$ and $f'\in L^1[\vartheta_1,\vartheta_2]$. If $|f'|$ is $s-$convex on $[\vartheta_1,\vartheta_2]$ for some fixed $s\in(0,1]$, then the following inequality holds,*

$$\left|R\left(f,Z\right)\right|\leq\sum_{i=0}^{n-1}\frac{17\left(x_{i+1}-x_{i}\right)^{2}}{60}\left(\frac{\left|f'\left(x_{i}\right)\right|+\left|f'\left(x_{i+1}\right)\right|}{2}\right).$$

Proof. Applying the Corollary 2.1 on the subinterval $[x_i,x_{i+1}]$, $(i=0,1,2,3,\ldots,n-1)$ of the partition Z and $\alpha=1$, $\beta(0)=\beta(1)=1$, we have

$$\begin{aligned}&\left|\frac{1}{15}\left(8f\left(\frac{3x_{i}+x_{i+1}}{4}\right)-f\left(\frac{x_{i}+x_{i+1}}{2}\right)+8f\left(\frac{x_{i}+3x_{i+1}}{4}\right)\right)-\frac{1}{x_{i+1}-x_{i}}\int_{x_{i}}^{x_{i+1}}f\left(x\right)dx\right|\\&\quad\leq\frac{17\left(x_{i+1}-x_{i}\right)}{60}\left(\frac{\left|f'\left(x_{i}\right)\right|+\left|f'\left(x_{i+1}\right)\right|}{2}\right).\end{aligned}\tag{8}$$

Multiplying both sides above inequality (8) with $(x_{i+1}-x_i)$ and then summing over i from 0 to $n-1$ and let that $|f'|$ is $s-$convex, by the triangle inequality, we obtained the desired result. □

5 Conclusions

In this paper, we established new identity for the corrected dual Simpson's type inequalities via Caputo–Fabrizio fractional integral operator. With the help of new derived identity for Caputo–Fabrizio fractional integral operator, the corrected dual Simpson's type inequalities are obtained for differentiable mapping whose absolute value are convex. These obtained inequalities will more effective when apply to the error estimates of the quadrature formula, and to special means. These findings may attract the intention of those researchers, who are working on fractional integral operators. Our technique is also plausible to give extensions for other fractional integral operators, e.g., $k-$Riemann–Liouville, Katugampola, conformable, and Atangana–Baleanu. Moreover, one can also extend these results in quantum calculus by exhibiting our method of arrangements of kernels for differentiable convex functions.

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Conflicts of Interest The authors declare no conflict of interest.

References

[1] T. Abdeljawad, S. Rashid, Z. Hammouch & Y. M. Chu (2020). Some new local fractional inequalities associated with generalized $(s,m)-$convex functions and applications. *Advances in Difference Equations*, *2020*(1), Article ID: 406. https://doi.org/10.1186/s13662-020-02865-w.

[2] T. Abdeljawad, S. Rashid, Z. Hammouch, İ. İşcan & Y. M. Chu (2020). Some new Simpson-type inequalities for generalized $p-$convex function on fractal sets with applications. *Advances in Difference Equations*, *2020*, Article ID: 496. https://doi.org/10.1186/s13662-020-02955-9.

[3] A. O. Akdemir, M. E. Özdemir, M. A. Ardıç & A. Yalçın (2017). Some new generalizations for GA-convex functions. *Filomat*, *31*(4), 1009–1016. http://dx.doi.org/10.2298/FIL1704009A.

[4] M. Alomari, M. Darus & S. S. Dragomir (2009). New inequalities of Simpson's type for $s-$convex functions with applications. *Research Report Collection*, *12*(4), 1–18.

[5] S. I. Butt, I. Javed, P. Agarwal & J. J. Nieto (2023). Newton–Simpson-type inequalities via majorization. *Journal of Inequalities and Applications*, *2023*(1), 16. https://doi.org/10.1186/s13660-023-02918-0.

[6] S. I. Butt, S. Yousaf, A. O. Akdemir & M. A. Dokuyucu (2021). New Hadamard-type integral inequalities via a general form of fractional integral operators. *Chaos, Solitons & Fractals*, *148*, Article ID: 111025. https://doi.org/10.1016/j.chaos.2021.111025.

[7] M. Caputo & M. Fabrizio (2015). A new definition of fractional derivative without singular kernel. *Progress in Fractional Differentiation & Applications*, *1*(2), 73–85.

[8] S. S. Dragomir, R. P. Agarwal & P. Cerone (2000). On Simpson's inequality and applications. *Journal of Inequalities and Applications*, *5*(6), 533–579.

[9] T. Du, H. Wang, M. A. Khan & Y. Zhang (2019). Certain integral inequalities considering generalized $m-$convexity on fractal sets and their applications. *Fractals*, *27*(7), Article ID: 1950117. https://doi.org/10.1142/S0218348X19501172.

[10] I. Franjić & J. Pecarić (2006). On corrected dual Euler–Simpson formulae. *Soochow Journal of Mathematics*, *32*(4), Article ID: 575.

[11] J. Hadamard (1893). Étude sur les propriétés des fonctions entières et en particulier d'une fonction considérée par Riemann. *Journal de Mathématiques Pures et Appliquées*, *9*, 171–215.

[12] F. Hezenci (2023). A note on fractional Simpson type inequalities for twice differentiable functions. *Mathematica Slovaca*, *73*(3), 675–686. https://doi.org/10.1515/ms-2023-0049.

[13] J. Hua, B. Y. Xi & F. Qi (2015). Some new inequalities of Simpson type for strongly $s-$convex functions. *Afrika Matematika*, *26*(5), 741–752. https://doi.org/10.1007/s13370-014-0242-2.

[14] H. Hudzik & L. Maligranda (1994). Some remarks on $s-$convex functions. *Aequationes Mathematicae*, *48*, 100–111. https://doi.org/10.1007/BF01837981.

[15] J. L. W. V. Jensen (1906). Sur les fonctions convexes et les inégalités entre les valeurs moyennes. *Acta Mathematica*, *30*(1), 175–193. https://doi.org/10.1007/BF02418571.

[16] A. Munir, A. Qayyum, S. S. Supadi, H. Budak & I. Faiz (2024). A study of improved error bounds for Simpson type inequality via fractional integral operator. *Filomat*, *38*(10), 3415–3427. https://doi.org/10.2298/FIL2410415M.

[17] J. Nie, J. Liu, J. Zhang & T. Du (2019). Estimation-type results on the $k-$fractional Simpson-type integral inequalities and applications. *Journal of Taibah University for Science*, *13*(1), 932–940. https://doi.org/10.1080/16583655.2019.1663574.

[18] M. Z. Sarıkaya, H. Budak & S. Erden (2019). On new inequalities of Simpson's type for generalized convex functions. *Korean Journal of Mathematics*, *27*(2), 277–293. https://doi.org/10.11568/kjm.2019.27.2.279.

[19] M. Z. Sarikaya, E. Set & M. E. Ozdemir (2010). On new inequalities of Simpson's type for $s-$convex functions. *Computers & Mathematics with Applications*, *60*(8), 2191–2199. https://doi.org/10.1016/j.camwa.2010.07.033.

[20] M. Z. Sarikaya & H. Yaldiz (2015). On Hermite–Hadamard type inequalities for $\phi-$convex functions via fractional integrals. *Malaysian Journal of Mathematical Sciences*, *9*(2), 243–258.

[21] E. Set, J. Choi & A. Gözpinar (2021). Hermite–Hadamard type inequalities involving nonlocal conformable fractional integrals. *Malaysian Journal of Mathematical Sciences*, *15*(1), 33–43.

[22] E. Set, M. Z. Sarikaya & N. Uygun (2017). On new inequalities of Simpson's type for generalized quasi convex functions. *Advances in Inequalities and Applications*, *2017*, Article ID: 3.

[23] M. Tariq, H. Ahmad, S. K. Sahoo, A. Kashuri, T. A. Nofal & C. H. Hsu (2022). Inequalities of Simpson–Mercer-type including Atangana–Baleanu fractional operators and their applications. *AIMS Mathematics*, *7*(8), 15159–15181. https://doi.org/10.3934/math.2022831.

[24] B. Y. Xi, R. F. Bai & F. Qi (2012). Hermite–Hadamard type inequalities for the $m-$ and $(\alpha, m)-$geometrically convex functions. *Aequationes Mathematicae*, *84*(3), 261–269. https://doi.org/10.1007/s00010-011-0114-x.

[25] T. Y. Zhang, A. P. Ji & F. Qi (2012). On integral inequalities of Hermite–Hadamard type for $s-$geometrically convex functions. *Abstract and Applied Analysis*, *2012*(1), Article ID: 560586. https://doi.org/10.1155/2012/560586.